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QUASISTEADY APPROACH FOR THERMAL ANALYSIS OF INSULATED STRUCTURES

R. SESHADRI
Syncrude Canada Ltd., Edmonton, Canada

and

A. V. G. KRISHNAYYA
Alberta Transportation, Edmonton, Canada

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Abstract—The quasisteady method of analysis has been utilized to determine the thaw or frost depth below heated or chilled insulated structures. Specifically, the method is applied to buried circular pipes, infinite strips and circular discs. For the case when the ground temperature is different from the phase change temperature, the solution is obtained by numerical integration of a quadrature. For the case when the ground temperature approaches the phase change temperature closed form solutions are obtained. The results presented in this paper should find use in the engineering design of structures in the colder regions of the world.

NOMENCLATURE

x, y, z, the rectangular coordinate system;
x_0, y_0, z_0, variables defining moving boundary;
t, time;
$P, \bar{P}, Q$, points on axis ($x > 0$, $y = z = 0$);
T_1, temperature (thawed zone);
T_2, temperature (frozen zone);
T_P, temperature (heated or chilled surface);
$\bar{T}_P$, temperature (insulation surface);
T_G, initial ground temperature;
T_0, phase change temperature;
S_1, surface of the heated areas;
$\bar{S}_1$, surface of the outside of the insulation;
S_0, moving surface separating the thawed and frozen zone;
C_1, volumetric heat capacity (thawed zone);
C_2, volumetric heat capacity (frozen zone);
K_1, thermal conductivity (thawed zone);
K_1^*, enhanced thermal conductivity (including convection);
K_2, thermal conductivity (frozen zone);
K_0, permeability of the ground;
L_1, latent heat (volumetric) of thawed zone;
n, outward normal;
l, characteristic length of the porous enclosure;
$x_{\bar{P}}$, depth evaluated at point $\bar{P}$;
x_Q, depth evaluated at point Q;
X, depth ratio;
$X_{\bar{P}}$, X evaluated at $\bar{P}$;
X_Q, X evaluated at Q;
R, radius of thaw or frost for a pipe buried in an infinite region;
$F(\ )$, temperature distribution function;
$F'(\ )$, derivative of $F(\ )$ with respect to ζ;
Ste, Stefan's number, $\dfrac{C_1(\bar{T}_P - T_0)}{L_1}$;
Da, Darcy number, $\dfrac{K_0}{l^2}$;
Ra, Rayleigh number, $\dfrac{(\bar{T}_P - T_0)gl^3}{T_0 \alpha \nu}$;
h_0, depth of burial of circular pipe;
s, thickness of insulation;
r_0, radius of buried pipe or of circular tank;
a, half width of strip;
g, acceleration due to gravity;
$G(\ )$, a known function defining relationship between $\bar{T}_P$ and T_P;
I_1, I_2, I_3, time factors for buried pipe, strip and circular tank, respectively.

Greek symbols

ζ, dummy variable for X;
∇, Laplacian;
θ_1, ratio of thermal conductivity of insulation to thermal conductivity of thawed soil;
μ, ratio of depth of burial of pipe to its radius;
δ, insulation thickness ratio (buried circular pipe);
δ_1, insulation thickness ratio (infinite strip);
δ_2, insulation thickness ratio (circular base);
α, thermal diffusivity;
ν, kinematic viscosity;
$\bar{\varepsilon}$, infinitesimal quantity;
ψ, a functional relationship.

INTRODUCTION

THE STUDY of the thermal regime in the ground, of heated or chilled structures, is of considerable importance in the colder regions of the world. In

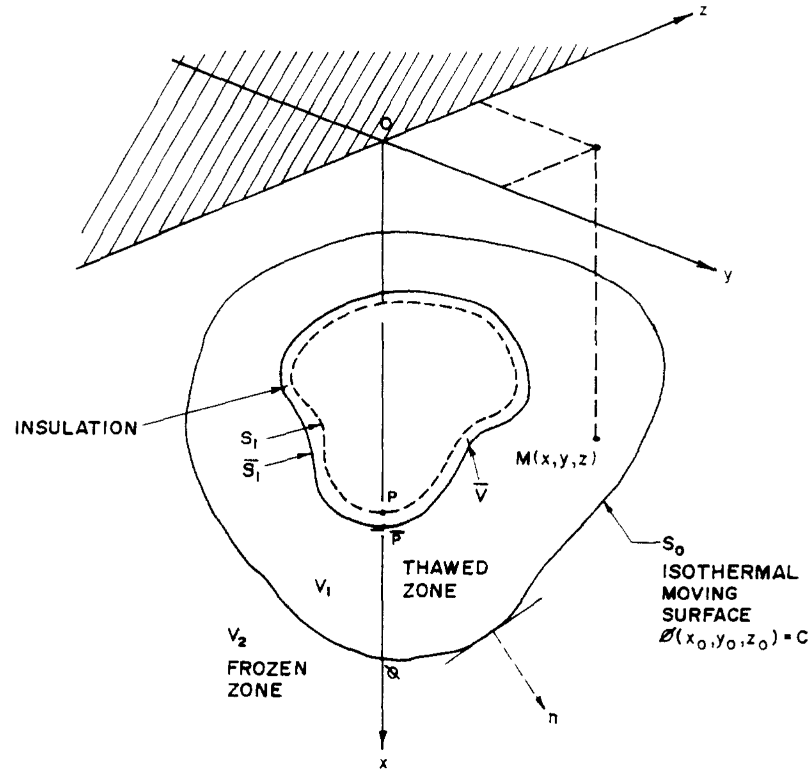

FIG. 1. The generalized two phase configuration.

such areas, the prediction of the extent of thawing or freezing induced by heated or chilled structures is relevant in engineering design and logistics. Specifically, the effects resulting from the erection of buildings; stripping of surface vegetation; construction of tank farms, buried water and sewer pipes, oil and gas pipelines, are of prime importance. Engineering designs are usually related to the determination of insulation requirement and emplacement of backfills as measures to minimize either thaw settlement or frost heave.

The work of Lachenbruch is relevant with respect to the study of the thermal regime below heated buildings [1] and around buried warm pipeline in permafrost [2]. Significant progress in numerical modelling of the thermal regime of buried pipes using finite element or variational methods have been made to date [3, 4]. However, such procedures are usually warranted only when the thermal configuration is complex. For preliminary engineering designs approximate methods for predicting the thermal regime may be effectively utilized. One such approximate method is the so called "quasisteady method" which is generally valid for cases where the latent heat of fusion is very large compared with the heat capacities [5]. Carslaw and Jaeger [6] have derived an approximate closed form quasisteady solution for thawing and freezing around a cylinder in an infinite domain. The same technique is used by Porkhayev [7] for estimating thaw depths below heated foundations. A good discussion relating to the quasisteady technique is given by Tsytovich [8].

In this paper, the quasisteady method has been applied to the problem of thawing or freezing below heated or chilled insulated structures, respectively. Also, some new closed form solutions for some practically useful geometries have been derived. These results should be of direct use in the engineering designs of the Arctic, Sub-Arctic and other colder regions of the world.

THE QUASISTEADY APPROACH

When moist ground thaws or freezes, a substantial portion of the heat is exchanged in effecting a phase change of the moisture in the ground. For soils with sufficient water content, the interface between the frozen and unfrozen zones usually moves very slowly. Consequently, the thermal regime at any given instant can be regarded as nearly steady. Thus, the transient nature of the process may be regarded as a continuous transition from one steady state to another [6–8].

The main requirement for the application of the quasisteady method of analysis are:

(a) The interface between the thawed and frozen zones of the moist ground is an isothermal surface of a steady temperature field.

(b) In the thawed and frozen zones, the temperature fields are described by the equations of steady state temperature field with the zone interface unsteady.

In this paper, the problem of thawing of frozen ground due to a heated structure is discussed.

However, the results are applicable to the converse problem. With reference to Fig. 1, S_1, is the surface on which a temperature is prescribed, corresponding to the boundary of the heated or chilled structure. $\bar{S}_1$, is the surface of the thermal insulation and S_0 is the isothermal moving surface representing the moving interface between the thawed and frozen zones. $\bar{V}$ is the zone occupied by the thermal insulation. V_1 is the thawed zone and V_2 is the frozen zone of the half-space.

The formal problem of thawing or freezing of the half-space based on heat conduction can be stated as follows:

The subscripts 1 and 2 refer to the thawed and frozen states, respectively. The temperature at a point $M(x, y, z)$ (see Fig. 1) at a given time $t (t \geqslant 0)$ is denoted by $T(x, y, z; t)$.

(a) For the thawed zone,

$$\nabla^2 T_1 = \frac{C_1}{K_1} \frac{\partial T_1}{\partial t}. \tag{1}$$

For the frozen zone,

$$\nabla^2 T_2 = \frac{C_2}{K_2} \frac{\partial T_2}{\partial t} \tag{2}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

C_1 and C_2 are the volumetric heat capacities and K_1 and K_2 are the thermal conductivities in the thawed and frozen zones, respectively.

(b) Initial condition

$$T_2(x, y, z; t = 0) = T_G \tag{3}$$

T_G is the initially uniform ground temperature.

(c) Fixed boundary conditions

$$T_2(x = 0, y, z; t) = T_G \tag{4}$$

at the ground surface

$$T_1(x, y, z; t) = \bar{T}_P \tag{5}$$

on the surface S_1

(d) Moving boundary conditions (Cauchy conditions)

$$T_1(x_0, y_0, z_0; t) = T_2(x_0, y_0, z_0; t) = T_0. \tag{6}$$

Points x_0, y_0, z_0 locate the isothermal moving surface S_0 corresponding to the phase change temperature T_0.

Also, on the moving surface S_0,

$$K_1 \frac{\partial T_1}{\partial n} - K_2 \frac{\partial T_2}{\partial n} = L_1 \frac{d\phi}{dt} \tag{7a}$$

where $\phi(x_0, y_0, z_0) = 0$, locates the isothermal moving surface, n is the outward normal on a given point on S_0, and $d\phi/dt$ is evaluated along the normal. L_1 is the volumetric latent heat.

The temperature at a given point on the surface $\bar{S}_1$ is related to the fixed temperature T_P prescribed on S_1, viz. $\bar{T}_P = G(T_P)$ where $G(\ )$ is a known function. This relationship can be established by equating the heat flow through the insulation to the heat flow into the thawed ground at the given point.

The "quasisteady assumption" requires that the moving boundary progresses very slowly as compared to the heat-conduction process in the thawed and frozen zones. This approximation is valid when the latent heat of the ground is large, viz. for a ground with considerable moisture. It is then reasonable to expect the approximation to be valid for situations when the Stefan's number, $C_1(\bar{T}_P - T_0)/L_1$, is small (Stefan's number, *Ste*, is the ratio of the sensible heat to the latent heat).

Since steady state conditions can be assumed in the thawed or frozen zones in a ground when $Ste \ll 1$ the temperature distribution in the thawed zone can be expressed as

$$T_1(x, y, z) = A_1 + B_1 F(x, y, z). \tag{8a}$$

Using potential theory [1, 6], the distribution for steady state thermal conditions, $F(x, y, z)$, can be determined for heat conduction in a homogeneous half space, for several configurations of the surface $\bar{S}_1$.

Using the boundary conditions, equations (5) and (6), the constants A_1 and B_1 can be evaluated. Consequently,

$$\frac{T_1(x, y, z) - T_0}{\bar{T}_P - T_0} = \frac{F(x, y, z) - F(x_0, y_0, z_0)}{1 - F(x_0, y_0, z_0)}. \tag{8b}$$

Also, by definition $F(\bar{S}_1) = 1$.

Similarly, the temperature distribution in the frozen region can be written as

$$T_2(x, y, z) = A_2 + B_2 F(x, y, z) \tag{9a}$$

where A_2 and B_2 are constants.

Using boundary conditions (6) and the condition $F(x \to \infty, y \to \infty, z \to \infty) = 0$, the temperature distribution can be written as:

$$\frac{T_2(x, y, z) - T_0}{T_G - T_0} = 1 - \frac{F(x, y, z)}{F(x_0, y_0, z_0)}. \tag{9b}$$

In many design problems the maximum extent of thaw or frost depth is required. For the specific geometries considered in this paper, it is assumed that the maximum thaw or frost occurs along the plane or axis of symmetry of the temperature field. It must be understood, however, that equation (7a) can be specialized for any direction. With reference to Fig. 1, simplification of the equation (9b) along the plane of symmetry $(x > 0, y = 0, z)$ or along the axis of symmetry $(x > 0, y = z = 0)$, i.e. for two and three dimensional problems, respectively, gives

$$K_1 \left. \frac{\partial T_1}{\partial x} \right|_{x_0} - K_2 \left. \frac{\partial T_2}{\partial x} \right|_{x_0} = L \frac{dx_0}{dt} \tag{7b}$$

where

$$\frac{T_1(x)-T_0}{\bar{T}_P-T_0}=\frac{F(x)-F(x_0)}{1-F(x_0)} \tag{8c}$$

and

$$\frac{T_2(x)-T_0}{T_G-T_0}=1-\frac{F(x)}{F(x_0)}. \tag{9c}$$

It is also assumed that the insulation thickness is small compared to a characteristic length and uniform around or along the heating surface, so that within the insulation steady state conditions apply. Integrating equation (7b) with the aid of equations (8c) and (9c), the following expression can be obtained.

$$\frac{K_1(T_P-T_0)t}{r_0^2L_1}=\int_{X_{\bar{P}}}^{X_Q}\frac{T_P-T_0}{\bar{T}_P(\zeta)-T_0}\cdot\frac{[F(\zeta)-1]}{\left[1-q(\bar{T}_P)\left\{\dfrac{F(\zeta)-1}{F(\zeta)}\right\}\right]}\cdot\frac{d\zeta}{F'(\zeta)} \tag{10}$$

where

$$X=\frac{x_0}{r_0};\quad q(\bar{T}_P)=\frac{K_2(T_G-T_0)}{K_1(\bar{T}_P(X)-T_0)}$$

and r_0 is a characteristic length.

X_P and X_Q can be interpreted to mean X at point $\bar{P}$ and X at point Q, respectively. When the ground temperature T_G approaches T_0 such that $T_G-T_0=\bar{\varepsilon}$ (infinitesimal quantity), $F(\zeta)$ can be still defined for the boundary valve problem for steady state conditions. However, if $T_G=T_0$, the boundary ($x=0$, y,z) has no effect on the temperature distribution in the thawed zone.

When the ground temperature approaches T_0, $q(\bar{T}_P)\to 0$ and equation (10) simplifies to:

$$\frac{K_1(T_P-T_0)t}{r_0^2L_1}=\int_{X_{\bar{P}}}^{X_Q}\frac{T_P-T_0}{\bar{T}_P(\zeta)-T_0}\cdot\left[\frac{F(\zeta)-1}{F'(\zeta)}\right]\cdot d\zeta. \tag{11}$$

In this paper, equation (11) has been integrated to give closed-form solutions for some practically useful geometries, while equation (10) is evaluated by a numerical integration provided $F(\zeta)$ is known. For the case when T_G-T_0 approaches $\bar{\varepsilon}$,

$$K_1\left.\frac{\partial T_1}{\partial x}\right|_{x_0}\gg K_2\left.\frac{\partial T_2}{\partial x}\right|_{x_0},$$

leading to equation (11). Also, if $T_G=T_0$, the shape function $F(\zeta)$ may have to be redefined, since the ground surface has no influence on the thaw bulb.

ASSUMPTIONS AND LIMITATIONS OF THE QUASISTEADY APPROACH

The prediction of the extent of thawing or freezing beneath heated or chilled structures is usually required in soils with moderate to high moisture content. Soils with low moisture content do not pose problems either with respect to thaw settlement or frost heave. It is therefore reasonable to obtain solutions for the situations when the Stefan number, *Ste*, is very small ($\ll 1$). The thaw-front would move slowly, and the transient problem can be regarded as a smooth transition from one steady state to the next.

With regard to the use of the shape-factors $F(x,y,z)$ in the quasisteady formulation there are two distinct considerations. Firstly when $T_G\neq T_0$, there is a flow of heat in the frozen zone, i.e. the surface ($x=0$, y,z) influences the temperature field in the frozen zone and hence the shape of the thaw-front. This is valid even when T_G-T_0 is an infinitesimal quantity, $\bar{\varepsilon}$. However, when T_G is exactly equal to T_0, there is no heat flow in the frozen zone and the shape of the thaw-front is determined by the surface $\bar{S}_1$. This exception in assuming $F(x,y,z)$ should be considered for every geometry of the surface, $\bar{S}_1$.

Other assumptions that have been made in the formulation of the quasisteady approach are as follows:

(a) The thickness of insulation is assumed to be small relative to the characteristic length in the problem, so that almost steady state conditions inside the insulation exist.

(b) The average surface temperature is equal to the average ground temperature.

(c) The temperature field $F(x,y,z)$ is assumed to have an axis of symmetry ($x>0$, $y=z=0$) for three dimensional problems, or a plane of symmetry ($x>0$, $y=0$, z) for two dimensional problems.

(d) The geothermal gradient is defined as the change in the temperature of the earth with depth and is usually expressed in degrees per unit depth. The geothermal heat flux, which is effectively the product of the geothermal gradient and the corresponding thermal conductivity, provides the basal boundary condition for the thermal calculations related to the ground temperature regime, especially in thermally sensitive regions of the world such as discontinuous and continuous permafrost areas. In the present paper, the geothermal heat flux is neglected.

TWO DIMENSIONAL ANALYSIS

Buried circular pipe

A circular pipe of radius r_i is buried at a depth of h_0 below the ground surface. If s is the thickness of insulation around the pipe, the external radius of the insulated pipe is $r_0=r_i+s$.

With reference to Fig. 2(a), the steady state temperature distribution can be expressed by the Forchheimer equation, [7],

$$T(x,y)=\bar{T}_P\frac{\ln\dfrac{y^2+[x+(h_0^2-r_0^2)^{1/2}]^2}{y^2+[x-(h_0^2-r_0^2)^{1/2}]^2}}{2\ln\left\{\dfrac{h_0}{r_0}+\left[\left(\dfrac{h_0}{r_0}\right)^2-1\right]^{1/2}\right\}} \tag{12a}$$

$$=\bar{T}_PF(x,y). \tag{12b}$$

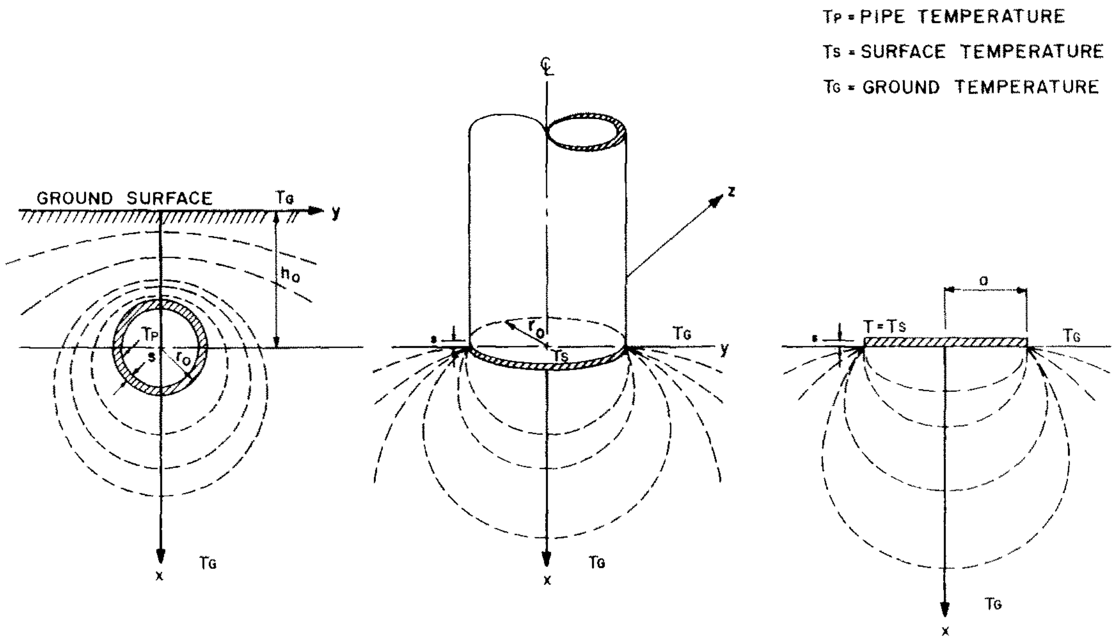

FIG. 2. Configuration for buried pipe, infinite strip and cylindrical vessels.

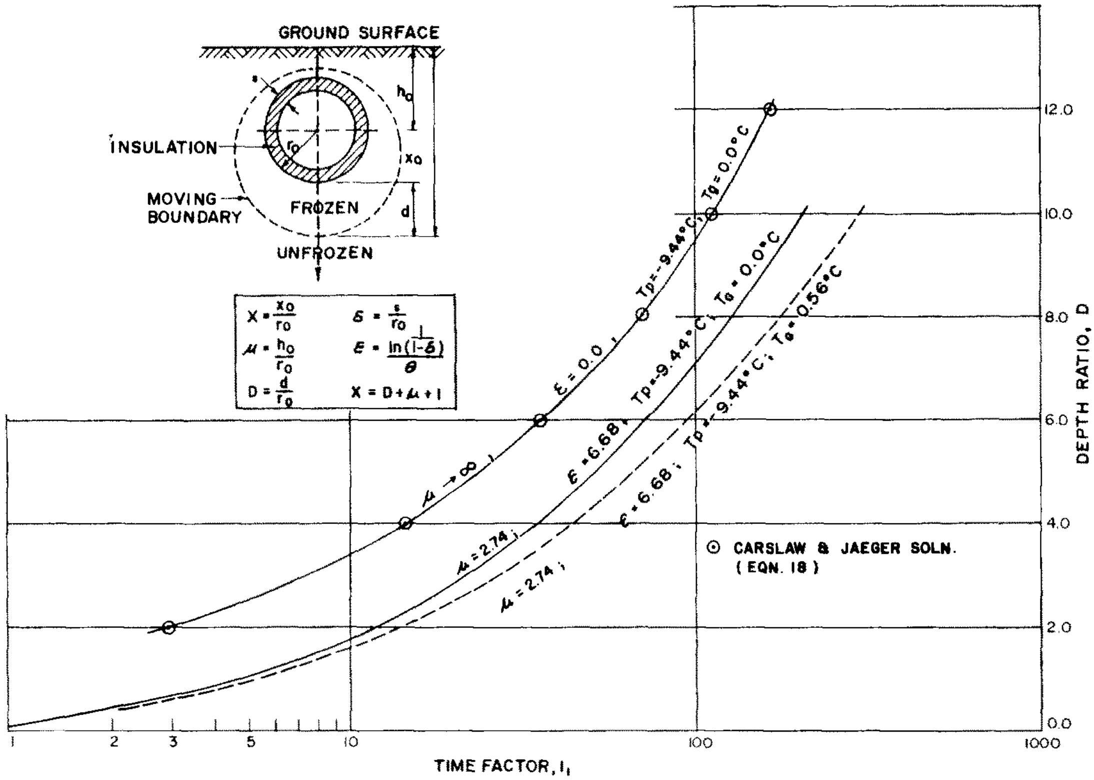

FIG. 3. Depth ratio vs time factor I_1 (buried insulated pipe).

For the plane of symmetry $(x; y = 0, z)$, the equation simplifies to

$$T(x, y=0) = \bar{T}_P \cdot \frac{\ln \dfrac{\zeta + (\mu^2 - 1)^{1/2}}{\zeta - (\mu^2 - 1)^{1/2}}}{\ln\left[\mu + (\mu^2 - 1)^{1/2}\right]} = \bar{T}_P F(\zeta) \qquad (13)$$

where

$$\zeta = \frac{x}{r_0}; \quad \theta_1 = \frac{\bar{K}}{K_1}; \quad \delta = \frac{s}{r_0}; \quad \mu = \frac{h_0}{r_0};$$

$\bar{K}$ and K_1 are the thermal conductivities of the insulation and thawed ground, respectively. For the specific case then $T_G = T_0$, the temperature distribution would be concentric about the pipe centre since the ground surface has no influence on the thaw-front. When the thaw-front touches the ground surface, the solution becomes invalid.

However, for $T_G < T_0$ the shape factor $F(\zeta)$, equation (13), can be assumed in the analysis.

By equating the heat flux per unit area going through the insulation to that going into the thawed zone at the point $\bar{P}$,

$$\frac{T_P - T_0}{T_P(X) - T_0} = 1 + \frac{\ln\left(\dfrac{1}{1-\delta}\right) F'(X_P)}{\theta[F(X) - 1]}. \tag{14}$$

Where

$$F'(X_P) = \frac{-2(\mu^2-1)^{1/2}}{\{(\zeta_P)^2 - \mu^2 + 1\} \ln[\mu + (\mu^2-1)^{1/2}]} \tag{15a}$$

$$\zeta_P = \mu + 1. \tag{15b}$$

The quadrature for the condition $T_G < T_0$ can be obtained by using equations (13), (14) and (10):

$$\frac{K_1(T_P - T_0)t}{r_0^2 L_1} = \int_{\mu+1}^{X_Q} \left[1 + \frac{\ln\left(\dfrac{1}{1-\delta}\right) F'(X_P)}{\theta_1\{F(X)-1\}}\right] \cdot \frac{\dfrac{\ln \dfrac{X + (\mu^2-1)^{1/2}}{X - (\mu^2-1)^{1/2}}}{\ln[\mu + (\mu^2-1)^{1/2}]} \cdot \dfrac{(X^2 - \mu^2 - 1)\ln[\mu + (\mu^2-1)^{1/2}]}{2(\mu^2-1)^{1/2}}\, \mathrm{d}X}{1 - q(X_P)\left\{1 - \dfrac{\ln[\mu + (\mu^2-1)^{1/2}]}{\ln \dfrac{X + (\mu^2-1)^{1/2}}{X - (\mu^2-1)^{1/2}}}\right\}}. \tag{16}$$

When the ground temperature $T_G \to T_0$ equation (16) can be integrated in closed form to give:

$$\begin{aligned}
\frac{K_1(T_P - T_0)t}{r_0^2 L_1} = \frac{(1-\mu^2)}{3} & \left[\left\{\left(\frac{X_Q}{(\mu^2-1)^{1/2}}\right)^3 \coth^{-1}\left(\frac{X_Q}{(\mu^2-1)^{1/2}}\right) - \left(\frac{\mu+1}{\mu-1}\right)^{3/2} \coth^{-1}[(\mu+1)/(\mu-1)]^{1/2}\right\}\right. \\
& \left. + \tfrac{1}{2}\left\{\ln \frac{\left(\dfrac{X_Q}{(\mu^2-1)^{1/2}}\right)^2 - 1}{\left(\dfrac{\mu+1}{\mu-1}\right) - 1} + \left(\frac{X_Q}{(\mu^2-1)^{1/2}}\right)^2 - \left(\frac{\mu+1}{\mu-1}\right)\right\}\right] \\
& + (\mu^2-1)\left[\frac{X_Q}{(\mu^2-1)^{1/2}} \coth^{-1}\left(\frac{X_Q}{(\mu^2-1)^{1/2}}\right) - \left(\frac{\mu+1}{\mu-1}\right)^{1/2} \coth^{-1}\left(\frac{\mu+1}{\mu-1}\right)^{1/2}\right. \\
& \left. + \tfrac{1}{2}\ln \frac{\left(\dfrac{X_Q}{(\mu^2-1)^{1/2}}\right)^2 - 1}{\left(\dfrac{\mu+1}{\mu-1}\right) - 1}\right] \frac{\ln[\mu + (\mu^2-1)^{1/2}]}{2(\mu^2-1)^{1/2}} \cdot \left\{\frac{\ln\left(\dfrac{1}{1-\delta}\right)}{\theta} \cdot F'(X_P) - 1\right\} \\
& \times \left[\left\{\frac{X_Q^3 - (\mu+1)^3}{3}\right\} - (\mu^2-1)\{X_Q - (\mu+1)\}\right].
\end{aligned} \tag{17}$$

Also,

$$\varepsilon = \frac{\ln\left(\dfrac{1}{1-\delta}\right)}{\theta_1}$$

can be defined as "insulation effectiveness ratio" and

$$I_1 = \frac{K_1(T_P - T_0)t}{r_0^2 L_1}$$

is the time factor.

The results equations (16) and (17) have use in the design of buried water, sewer, oil and gas pipes.

The progress of the frost front with time is plotted in Fig. 3, for a buried pipe with the following data:

Diameter of the pipe is 1.067 m (42 in); thickness of styrofoam insulation is 5.08×10^{-2} m (2 in); depth of burial of pipe is 1.60 m (5.25 ft); temperature of chilled gas is -9.44°C (15°F); average ground temperature 0°C (32°F) for case 1 and 0.56°C (33°F) for case 2: thermal conductivity of insulation is 2.94×10^{-2} W/m°K (0.017 Btu/h°F); thermal conductivities for a frozen and unfrozen ground and 2.16 W/m°K (1.25 Btu/h ft°F) and 1.56 W/m°K (0.90 Btu/h ft°F), respectively. The latent heat for wet ground, L_1, is 2.22×10^8 J/m³ (5950.0 Btu/ft³). When the pipe is buried at large depths, the influence of the ground surface is negligible and the isotherms would be circular and symmetric about the pipe axis. For the pipe with insulation

$$I_1 = 2\left(\frac{R}{r_0}\right)^2 \ln\left(\frac{R}{r_0}\right) - \left(\frac{R}{r_0}\right)^2 + 1 + \frac{2}{\theta}\left[\left(\frac{R}{r_0}\right)^2 - 1\right] \ln\left(\frac{1}{1-\delta}\right). \tag{18a}$$

When there is no insulation on the pipe, the result is given by Carslaw and Jaeger [6], viz.

$$I_1 = 2\left(\frac{R}{r_0}\right)^2 \ln\left(\frac{R}{r_0}\right) - \left(\frac{R}{r_0}\right)^2 + 1. \tag{18b}$$

A comparison between results (17) and (18b) is plotted in Fig. 3.

An infinitely long strip

Consider an infinitely long strip of width $2a$. Below the strip is an insulation of thickness t. With reference to Fig. 2, the steady state temperature

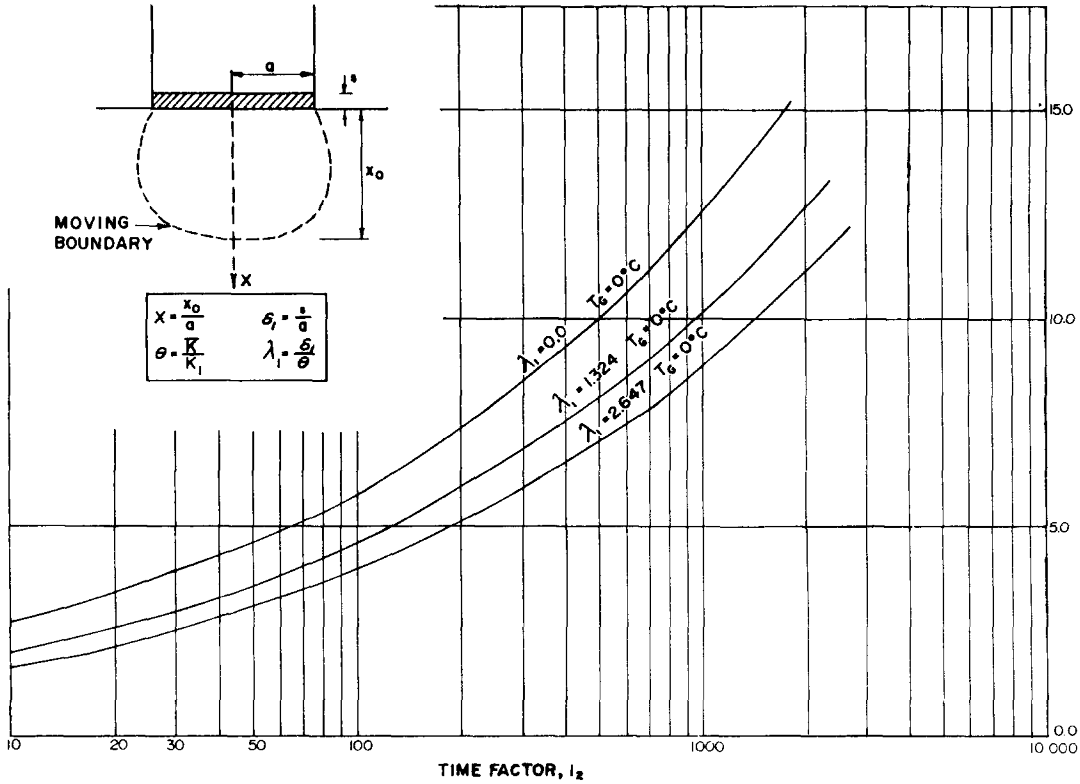

FIG. 4. Depth ratio vs time factor I_2 (infinite strip).

distribution for a half space in this case is given by

$$T(x,y) = \frac{\overline{T}_P}{\pi}\left(\arctan\frac{a+y}{x} + \arctan\frac{a-y}{x}\right) \quad (19a)$$

$$= \frac{\overline{T}_P}{\pi} F(x,y). \quad (19b)$$

Defining

$$\zeta = \frac{x}{a}, \quad \delta_1 = \frac{s}{a}, \quad \theta_1 = \frac{\bar{K}}{K_1}$$

the temperature distribution for the plane of symmetry (x, $y = 0$, z) is

$$T(\zeta) = \frac{2\overline{T}_P}{\pi}\arctan\left(\frac{1}{\zeta}\right). \quad (20)$$

The heat flux balance at a point such as P is given by

$$\frac{T_P - T_0}{\overline{T}_P(X) - T_0} = 1 + \frac{\delta_1}{\theta_1}\cdot\frac{F'(X_{\bar{P}})}{F(X)-1}. \quad (21)$$

The solution for $q \to 0$, viz. $T_G \to T_0$,

$$\frac{K_1(T_P - T_0)t}{a^2L_1} = \left(\frac{\pi}{2} + \frac{\delta_1}{\theta}\right)\left(\frac{X_Q^3}{3} + X_Q\right) - \left(\frac{X_Q^3}{3} + X_Q\right)\arctan\left(\frac{1}{X_Q}\right) - \frac{X_Q^2}{6} - \tfrac{1}{3}\ln\frac{1}{(1+X_Q^2)^{1/2}} - \tfrac{1}{2}\ln(1+X_Q^2). \quad (22)$$

For $T_G < T_0$ the quadrature becomes

$$\frac{K_1(T_P - T_0)t}{a^2L_1} = \frac{\pi}{2}\int_0^{X_Q}\left[1 + \frac{\delta_1/\theta_1\cdot F'(X_{\bar{P}})}{F(X)-1}\right] \cdot \frac{\left\{1 - \frac{2}{\pi}\arctan\left(\frac{1}{X}\right)\right\}\cdot(X^2+1)\cdot \mathrm{d}X}{\left[1 - q(X)\left\{1 - \frac{1}{\frac{2}{\pi}\arctan\left(\frac{1}{X}\right)}\right\}\right]}. \quad (23)$$

Consider as an example a strip of half-width equal to 3.05 m (10 ft) which is at a temperature of 15.55°C (60°F). The bottom of the strip is insulated with styrofoam ($K = 2.94 \times 10^{-2}$ W/mK or 0.017 Btu/h ft °F). The ground has the following thermal properties: $K_1 = 1.56$ W/m K (0.900 Btu/h ft °F), $K_2 = 2.16$ W/m K (1,250 Btu/h ft °F), $L_1 = 1.34 \times 10^8$ J/m³ (3,600 Btu/ft³). The rate of thaw is plotted in Figs. 4 and 5 for different insulation thicknesses and ground temperatures. Figure 5 is a plot of the depth ratio X against the time factor

$$I_2\left(= \frac{K_1(T_P - T_0)t}{a^2L_1}\right)$$

for different values of $\lambda_1 = \delta_1/\theta_1$. An increase in λ_1 indicates either an increase in insulation thickness of a decrease in insulation thermal conductivity. λ_1 is the "insulation effectiveness ratio" for this case.

Equations (22) and (23) can be applied to designs pertaining to strip footings or long rectangular areas.

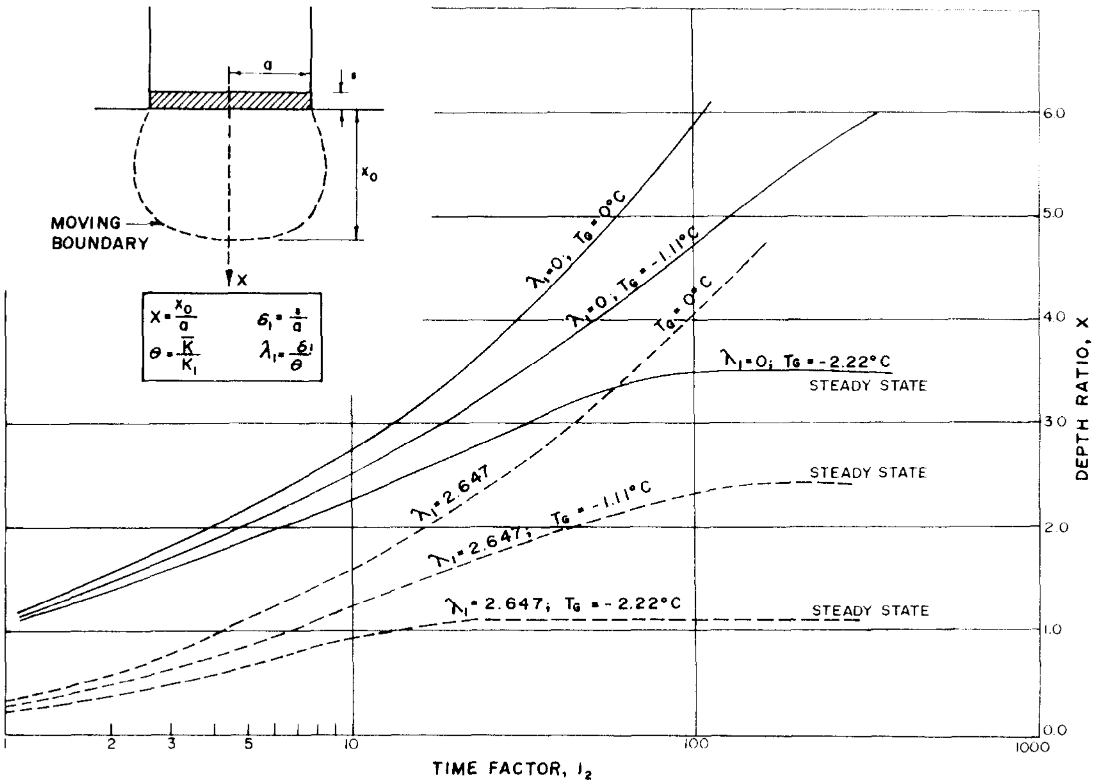

FIG. 5. Depth ratio vs time factor (infinite strip).

THREE DIMENSIONAL ANALYSIS

Tank with a circular base

Consider a tank with a circular bottom with a radius, r_0, and with an insulation of thickness t and thermal conductivity $\bar{K}$.

$$\text{Let } \zeta = \frac{x}{r_0},\ X = \frac{x_0}{r_0},\ \delta_2 = \frac{s}{r_0},\ \theta_1 = \frac{\bar{K}}{K_1}.$$

The expression for steady state temperature along the centerline can be written as [1].

$$F(\zeta) = 1 - \frac{\zeta}{(\zeta^2+1)^{1/2}}. \tag{24}$$

Equating heat flux at a point such as P.

$$\frac{T_P - T_0}{\bar{T}_P(X) - T_0} = 1 + \frac{\delta_2}{\theta_1}\frac{(X^2+1)^{1/2}}{X}. \tag{25}$$

Equations (24), (25) and (10) can be combined to give the quadrature for $T_G < T_0$ as in earlier cases.

For the case when $T_G \to 0$, the closed form result can be written as:

$$\frac{K_1(T_P - T_0)t}{r_0^2 L_1} = \left(\frac{X_Q^2}{2} + \frac{X_Q^4}{4}\right) + \frac{\delta_2}{4\theta_1}\,[X_Q[(X_Q^2+1)^3]^{1/2} + \tfrac{3}{2}\{X_Q(X_Q^2+1)^{1/2} + \ln(X_Q + [X_Q^2+1]^{1/2})\}]. \tag{26}$$

The analysis presented in this section has use in the design of tank pads. For a circular tank of radius 10 ft and the same thermal data as for the previous cases, the results are plotted in Figs. 6 and 7.

CONVECTIVE HEAT TRANSFER IN THE GROUND

In addition to heat transfer in the ground by conduction, the contribution due to convection can be significant under certain conditions. This is particularly true of liquids [10]. However, for a porous medium such as the ground, the effect of convection would be significantly reduced. There are two important aspects with regard to the effect of convection in a porous medium:

1. Criteria for the onset of convection.
2. Determination of correction factors that would account for the effect of convection in the conduction-model.

Chan *et al.* [11] have considered the problem of natural convection in an enclosed porous media with rectangular boundaries. Through the incorporation of the so called "enhanced thermal conductivity", (K_1^*), they include the effect of convection in the heat-conduction model, and a general relationship is proposed as follows:

$$\frac{K_1^*}{K_1} = \psi(Da, Ra, \Gamma) \tag{27}$$

where K_1^* and K_1 are enhanced and conventional thermal conductivities, respectively; "Da" is the Darcy number and "Ra" is the Rayleigh number. Γ is the shape-parameter dependent on the geometry of the porous enclosure. Further, $Da = K_0/l^2$, where K_0 = permeability of the ground; l = characteristic length; and $Ra = (\bar{T}_P - T_0)gl^3/T_0\alpha\nu$, where

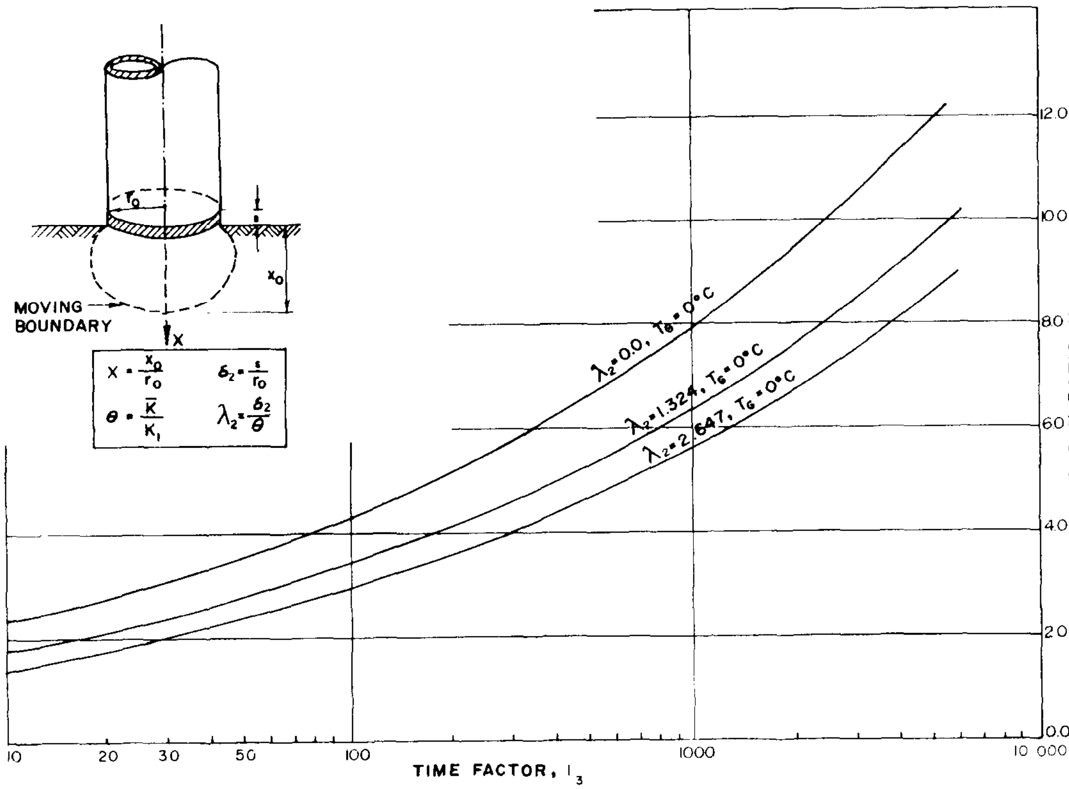

FIG. 6. Depth ratio vs time factor (circular area).

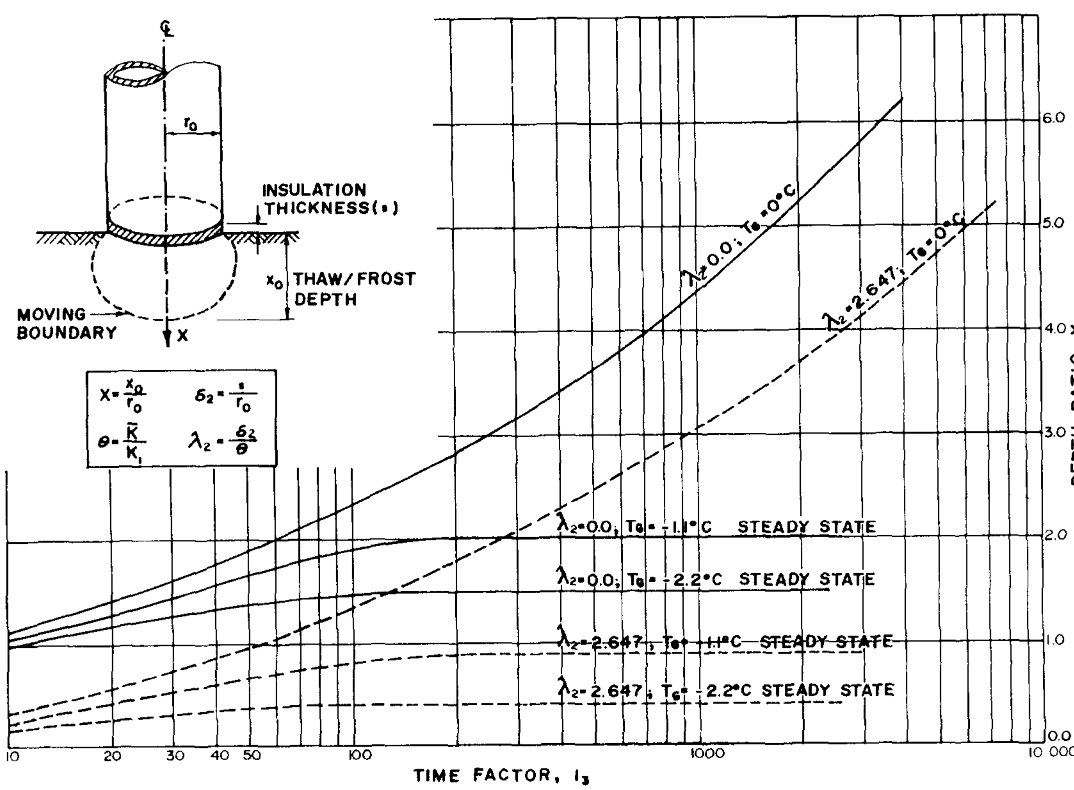

FIG. 7. Depth ratio vs time factor I_3 (circular area, different λ, T_0).

α = thermal diffusivity (ft^2/h) and ν = kinematic viscosity (lb/ft h).

The criterion for the onset of convection currents in a porous medium is obtained by the determination of the breakdown of stability of a layer of fluid subject to a temperature gradient [12].

For specific configurations such as buried pipe etc., the effect of convection is complex. However, approximations using solutions for simple boundaries of the porous enclosure [11] would give a reasonable estimate of the convective effects. From a practical point of view, the design of structures based on the conduction model without convective effects, would be conservative. The effect of natural convection is to underestimate the flow of heat along $\bar{P}Q$ (see Fig. 1) for the thawing problem.

CONCLUSION

The quasisteady analysis is applied to several useful geometries of heated and chilled structures in order to determine the movement with respect to time of the interface between the unfrozen and frozen regions below the ground surface.

For specific geometries such as buried pipe, strip footing and tanks with circular base, closed form solutions are obtained, when the ground temperature approaches the phase change temperature. When the ground temperature is different from phase change temperature, the solutions are obtained using numerical integration. Some results are compared with known closed form solutions and the comparison is encouraging. The results presented in this paper have numerous applications in the design of structures in the colder regions of the world.

The quasisteady method can be applied to cases where the steady state temperature profiles in the ground below heated or chilled structures for other geometries are known. Further work to include the effect of convection for the specific geometries of heating or cooling surfaces in the porous ground would be useful.

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APPROCHE EN REGIME QUASI STATIQUE POUR L'ANALYSE THERMIQUE DE STRUCTURES ISOLEES

Résumé—La méthode quasi statique d'analyse a été utilisée pour déterminer la profondeur de gel ou de dégel au dessous d'une structure chauffée ou refroidie. La méthode est appliquée à des conduites circulaires enterrées, à des bandes infinies et des disques circulaires. Pour le cas où la température du sol est différente de la température de changement de phase, la solution est obtenue par l'intégration numérique. Dans le cas où la température du sol approche la température de changement de phase, on obtient des solutions analytiques.

Les résultats présentés peuvent trouver une utilisation dans les projets d'ingénierie pour les régions froides de la terre.

QUASISTATIONÄRES VORGEHEN BEI DER THERMISCHEN BERECHNUNG ISOLIERTER STRUKTUREN

Zusammenfassung—Das quasistationäre Rechenverfahren wurde verwendet, um die Tau- oder Frosttiefe unter geheizten oder gekühlten isolierten Strukturen zu bestimmen. Insbesondere wird die Methode auf vergrabene Kreisrohre, unendliche Streifen und kreisförmige Platten angewendet. Im Falle, daß sich die Bodentemperatur von der Phasenänderungstemperatur unterscheidet, wird die Lösung durch numerische Integration gewonnen. Für den Fall, daß die Bodentemperatur sich der Phasenänderungstemperatur nähert, werden geschlossene Lösungen erhalten. Die in dieser Arbeit vorgelegten Ergebnisse sollten Anwendung beim bautechnischen Entwurf von Strukturen in den kälteren Regionen der Erde finden.

КВАЗИСТАЦИОНАРНЫЙ МЕТОД РАСЧЁТА ТЕПЛОВЫХ ХАРАКТЕРИСТИК ИЗОЛИРОВАННЫХ СТРОИТЕЛЬНЫХ ЭЛЕМЕНТОВ

Аннотация — Используется квазистационарный метод определения глубины оттаивания или промерзания ниже нагреваемого или охлаждаемого изолированного строительного элемента. Этот метод, в частности, применяется к заглубленным круглым трубам, бесконечным полосам и круглым дискам. При отличии температуры основания от температуры фазового перехода решение получается численным интегрированием. При близости этих температур решение получается в замкнутой форме. Результаты анализа могут найти применение в инженерных расчётах конструкций, предназначенных для холодных районов земного шара.